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SOLUTION OF THE EXTERIOR AND INTERIOR DIRICHLET PROBLEM OF POTENTIAL THEORY IN A MULTIPLY CONNECTED DOMAIN

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Through use of a complement to the solution of a heat conduction boundary value problem of Dirichlet type (presented classically in the form of a double layer potential) we obtain by means of simple sources singular integral equations (SIE) for exterior and interior multiply connected domains. Algorithms and a computer program were developed to obtain a numerical solution of the SIE.

In considering thermal problems of Dirichlet type by method of the potential (temperature T is a harmonic function and is subject to the equation of Laplace) two traditional methods are employed: classical (nondirect) and nonclassical (direct).

The classical method consists in seeking a solution in the form of a double layer potential:

$$T = \oint_S \varkappa(y) \frac{\cos\varphi}{r^2} dS_y. \tag{1}$$

Its limiting value at points of boundary S of domain V is equated to the given function and we obtain the following integral equation:

$$T(x_S) = 2\pi\varkappa(x_S)\eta + \text{v. p.} \oint_S \varkappa(y) \frac{\cos\varphi}{r^2} dS_y. \tag{2}$$

Here $T(x_S)$ is a given value of the function on boundary S of domain V; κ is the density of the double layer potential; φ is the angle between vector $r = |y - x|$ and the exterior normal n_y to S at the integration point y; $\eta = 1$ for the inner limit; $\eta = 0$ for the direct value; and $\eta = -1$ for the outer limit; v. p. indicates principal value of the Cauchy-type integral.

This method is used, however, only in the case of an interior simply connected domain [1]. For an exterior domain (even a simply connected one) it is not a suitable method. Actually the double layer potential can only represent the temperature of the exterior domain partially. If the temperature is split up into two components, a constant component and a variable component, $T = T^{(m)} + T^{(v)}$, where $T^{(m)}$ is the mean value, the influence of the mean temperature $T^{(m)}$ is then not taken into account by the double layer potential. In addition, for a simply connected exterior domain even a variable temperature field cannot be represented by a double layer potential if the sources are distributed uniformly over the boundary surface ($\kappa(y) = \text{const}$):

$$T = \varkappa \oint_S d\omega_S = \varkappa \oint_S \frac{\cos\varphi}{r^2} dS = \varkappa\omega_S \equiv 0 \tag{3}$$

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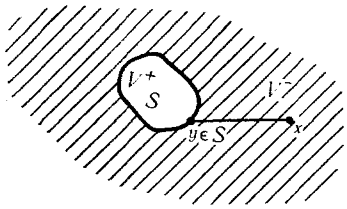

Fig. 1

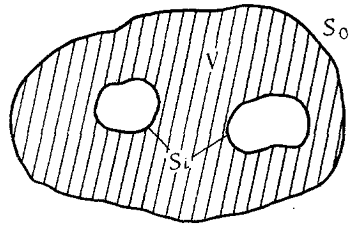

Fig. 2

Fig. 1. Definition of solid angle ω_S for surface S of exterior region.

Fig. 2. Scheme for isolating parts of space enclosed by surface S_0.

(the solid angle ω_S of the whole surface S is equal to zero when point x is located in the exterior region V^- (Fig. 1) and $T \equiv 0$). In the case of a multiply connected region V^- the situation remains unchanged so long as point $x \in V^-$.

We consider the direct method, i.e., when the solution is represented by Green's formula:

$$T = \frac{1}{4\pi} \oint_S \left(\frac{dT}{dn_y} \frac{1}{r} + T_S \frac{\cos\varphi}{r^2} \right) dS_y, \tag{4}$$

here T_S is the value of the temperature on the boundary surface S; dT/dn_y is the value of the normal derivative of the temperature.

In region V^+ let us assume there is a hot liquid with temperature T. Then the flow of heat from the liquid into the surrounding solid body [3] is given by

$$q = -\lambda \frac{dT}{dn}, \tag{5}$$

where λ is the heat transfer coefficient. The flow q may turn out to be the same at all points of the surface S, for example, owing to symmetry of the surface S itself. Under such boundary conditions the double layer potential in relation (4) will be equal to zero and the temperature is expressed thus:

$$T = \frac{1}{4\pi} \frac{dT}{dn} \oint_S \frac{1}{r} dS. \tag{6}$$

We apply a theorem relating to the mean value of the integral in Eq. (6). At a sufficiently large distance r_δ we have the formula

$$T \simeq \frac{1}{4\pi} \frac{dT}{dn} S \frac{1}{r_\delta} = -\frac{qS}{4\pi\lambda} \frac{1}{r_\delta} = -\frac{Q}{4\pi\lambda} \frac{1}{r_\delta}, \tag{7}$$

where Q is the total thermal flux through surface S, or Q = qS is the output of all the thermal sources present in region V^+. Expression (6) or (7) is the defining expression for temperature T in the exterior region V^- since an arbitrary constant temperature field $T^{(m)}$ = const (T must be a regular function at infinity, i.e., $T_\infty = 0$) cannot be added to the calculated value and the direct formulation yields the specified solution.

We consider now, instead of the infinite region V^-, a finite region V, obtained by isolating from V^- a portion of the space (see Fig. 2) enclosed by surface S_0: $V \to V^-$ as $S_0 \to \infty$.

In the classical approach we obtain, according to formula (3), a constant term for the enveloping surface S_0, equal to the temperature T_∞ at infinity:

$$T = T_\infty = \text{const} = 4\pi\varkappa_{S_0}. \tag{8}$$

In the direct approach we also obtain a constant term for T:

$$\frac{1}{4\pi} \oint_{S_0} T_\infty d\omega_0 = \frac{1}{4\pi} T_\infty \omega_{S_0} = T_\infty. \tag{9}$$

The potential of a simple layer for the enveloping surface s_0 in relation (4) is equal to zero, since $dT/dn = C/r_\infty^2 \to 0$, and the temperature, by formula (7), will be equal to zero,

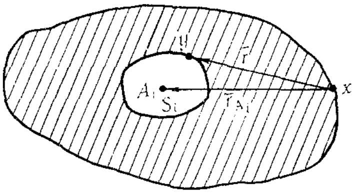

Fig. 3. Determination of strength of source A_i.

i.e., introduction of the enveloping surface corroborates uniqueness of the direct method solution. But for the classical approach, introduction of an enveloping surface still does not remedy the form of solution (1) since the mean value $\varkappa$ of densities of potentials on S_i does not contribute to the value of temperature T.

Using a device known in the theory of harmonic functions, we add to solution (1) simple sources of strength A_i, present inside surface S_i. Then

$$T = T_\infty + \oint_{S_i} \varkappa \frac{\cos\varphi}{r^2} dS + \sum_i \frac{A_i}{r_{A_i}}, \tag{10}$$

where r_{A_i} is the distance to a source (see Fig. 3).

Let us assume we have a surface S_i. We are required to calculate strength of the source starting from the given boundary conditions. In the case of a single surface, instead of relation (10), we write

$$T = T_\infty + \oint_S \varkappa d\omega + \frac{A}{r_A}. \tag{11}$$

Ideally, through point x we draw surface S_x (see Fig. 3; S_x is not to be confused with $S_0 \to \infty$, which yields the term T_∞). The mean value of temperature T on S_x is determined from the condition (theorem concerning the mean) given in [2]:

$$T^{(m)} = \frac{1}{S_x} \oint_{S_x} T_{S_x} dS_x. \tag{12}$$

Into relation (12) we substitute expression (11):

$$T^{(m)} = T_\infty + \frac{1}{S_x} A \oint_{S_x} \frac{dS_x}{r_A}. \tag{13}$$

We direct surface S_x towards the boundary surface S. Equation (13) here remains valid, but in the limit $T^{(m)}$ will be equal to the mean value of the temperature $T_S = f$, given in the Dirichlet problem on S:

$$T^{(m)} = \frac{1}{S} \oint_S f dS. \tag{14}$$

In the limit formula (13) then takes the form

$$\frac{1}{S} \oint_S f dS - T_\infty = A \oint_S \frac{dS}{r_A} \tag{15}$$

and the source strength is

$$A = \frac{1}{\oint_S \frac{dS}{r_A}} \left(\frac{1}{S} \oint_S f dS - T_\infty \right), \quad S = \oint_S dS. \tag{16}$$

If the number of surfaces is n, relation (16) may then be applied on each of them and makes it possible to calculate all the strengths A_i ($i = 1, 2, \ldots, n$). The limiting expression for the temperature as point x tends towards a point of an arbitrary one of the surfaces S_i may be written

$$T_s = T_\infty - 2\pi\varkappa(x) + \text{v. p.} \oint_S \varkappa(y) \frac{\cos\varphi}{r^2} dS_y + \sum_{i=1}^{n} \frac{A_i}{r_{A_i}}. \tag{17}$$

Equating expression (17) to the given function F, we obtain an integral equation for κ:

$$F = T_\infty - 2\pi\varkappa(x) + \text{v. p.} \oint_S \varkappa(y) \frac{\cos\varphi}{r^2} dS_y + \sum_{i=1}^{n} \frac{A_i}{r_{A_i}}. \quad (18)$$

Solving integral equation (18) for $\kappa(y)$, we determine, according to formula (10), temperature T at an arbitrary point of the region V^- being considered. We also apply this approach to the case of an interior multiply connected region V^+.

Denoting the enveloping surface of the k + 1 connected region by S_e, we require that at boundary points of surface S_k:

$$T_{k+1}^{(m)} = \frac{1}{S_{k+1}} \sum_{i=1}^{n} A_i \oint_{S_k} \frac{dS_k}{r_{A_i}} + 4\pi\varkappa_e^{(m)}; \quad (19)$$

this suggests a Dirichlet condition for $T^{(m)}$. Here $\kappa_e^{(m)}$ is the mean density of the distribution of sources on S_e. Taking relation (14) into account, we determine constant A from relation (20).

Substituting expression for the temperature

$$T = \oint_{S_{k+1}} \varkappa(y) \frac{\cos\varphi}{r^2} dS_y + \sum_{i=1}^{n} \frac{A_i}{r_{A_i}} \quad (20)$$

into the Dirichlet-type boundary conditions, we obtain an integral equation for the heat conduction boundary value problem for the interior multiply connected domain:

$$2\pi\varkappa(x_S) + \text{v. p.} \oint_{S_{k+1}} \varkappa(y) \frac{\cos\varphi}{r^2} dS_y + \sum_{i=1}^{n} \frac{A_i}{r_{A_i}} = F. \quad (21)$$

Solving Eq. (21) for κ, we determine the temperature in region V^+ from formula (20).

Thus, by adding simple sources to the solutions (1), (20), we manage to avoid a deficiency in the classical method for solving a thermal Dirichlet problem for exterior and interior multiply connected domains. The resulting formulas (10), (18), (20), (21) make it possible to solve three-dimensional problems (in [4, 5] integral equations are obtained for two-dimensional and axially-symmetric problems).

Algorithms were developed for obtaining numerical solutions on a computer. Integrals were calculated using Gaussian-type quadrature formulas with an even number of nodes; singular integrals were handled with the aid of cubature formulas [6]. To calculate elliptic integrals in the axially-symmetric case we used the Lashchenov formulas. Reliability of our formulas and the high accuracy of the algorithm were confirmed by numerically solving test problems on a computer, namely, boundary value problems for a hollow ball, an infinite cylinder, a cylinder of finite length, and others. Thus, using the programs developed here, solutions can be obtained for two-dimensional, axially-symmetric, and three-dimensional boundary value problems for Dirichlet-type heat conduction for piecewise-smooth exterior and interior multiply connected regions.

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